# Yiddish of the Day

"mekhaye" = מחיה

a great pleasure =

"covid" (lol) = כּבֿוד

honor/privildge =

# Lecture 12 - Spectral Theorem

## Last time

Def: $V$ innerproduct space $T : V \to V$ linear.

The **adjoint** of $T$ denoted $T^*$

is the ! operator $T^* : V \to V$ st

$$\langle Tv, w \rangle = \langle v, T^*w \rangle \quad \forall v, w \in V$$

We are interested in special transformations

**Def:** $T: V \to V$ linear. Then we say $T$ is

1) normal if $T \circ T^* = T^* \circ T$

2) self-adjoint (Hermitian) if $T = T^*$

**Prop:** For any $T: V \to V$ the operators

$S_1 = T \circ T^*$ and $S_2 = T^* \circ T$ are self-adjoint

Pf) 1) $(T \circ T^*)^* \overset{\text{last class}}{=} T^{**} \circ T^* = T \circ T^*$

2) $(T^* \circ T)^* = T^* \circ T^{**} = T^* \circ T$

<u>Prop</u>: Let $T: V \to V$ be <u>self-adjoint</u> then any <u>eigenvalues</u> of $T$ are <u>real</u>

Pf) Suppose $\lambda$ is an eigenvalue w/ eigenvector $v$

Compute $\langle Tv, v \rangle = \langle v, Tv \rangle = \langle v, \lambda v \rangle = \bar{\lambda} \langle v, v \rangle$

$\parallel$

$$\langle \lambda v, v \rangle = \lambda \langle v, v \rangle$$

$\Rightarrow \lambda \langle v,v \rangle = \bar{\lambda} \langle v,v \rangle$ and since $\langle v,v \rangle \neq 0 \Rightarrow \lambda = \bar{\lambda}$ $\boxed{\text{☺}}$

**Prop:** $T: V \to V$ self-adjoint and $\mathcal{B}$ an ON basis. Then

$$[T]_{\mathcal{B}} = \overline{[T]}^{tr} \qquad \text{ex) } \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}$$

Pf) Exercise ☺

(Consequence of HW question)

**Prop:** $T, S : V \to V$ self-adjoint. Then

$S+T$ is too

[ Q: Is $(T \circ S)$? ]

Pf) $(S+T)^* = S^* + T^* = S + T$

think about this

Thm: Spectral Thm for normal operators.

Let $T$ be normal operator. Then $\exists$ ON basis $B$ of eigenvectors of $T$

$$[T]_B = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Cor: Spectral Thm for Self-adjoint operators.

Let $T: V \to V$ be self-adjoint then there exists a basis of ON eigenvectors for $T$, with all real eigenvalues

(So, wrt this basis is $[T]_B = \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix}$ with $r_i \in \mathbb{R}$)

Rmk: True in $\infty$-dim

$\rightsquigarrow$ much harder proof

$\rightsquigarrow$ essential in QM.

Operators are "Hermitian" (= self-adjoint)

The basis of eigenvectors are the

observables and the eigenvalues

are the measurements

ex) $\hat{H}\psi = E\psi$

$\left( \hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + V \right)$ $E$ = energy

**Def:** We call an operator <u>orthogonal</u> if (usually over $\mathbb{R}$)

$T^* = \underline{T^{-1}}$

(sometimes called <u>unitary</u> - usually over $\mathbb{C}$)

(sometimes called "linear isometries")

Thm: TFAE (the following are equivalent)

(1) $T$ is orthogonal (ie $T^* = T^{-1}$)

2) $\langle Tx, Ty \rangle = \langle x, y \rangle \quad \forall\, x, y \in V$

3) $\|Tx\| = \|x\| \quad \forall x \in V$

((2) = preserves "angles"

(3) = preserves "distances")

**Prop**: $T: V \to V$ orthogonal. Then any eigenvector of $T$ has eigenvalue with $|\lambda| =$ 1 ($\lambda^2 = 1$ or $\lambda\bar{\lambda} = 1$)

Pf) Suppose $v$ is an eigenvector.

$$\langle Tv, Tv \rangle = \langle v, v \rangle$$

$$\| \quad\quad\quad\quad \implies \lambda\bar{\lambda} = 1$$

$$\langle \lambda v, \lambda v \rangle$$

$$\lambda\bar{\lambda} \langle v, v \rangle$$

**Def:** A $n \times n$ real matrix. Say $A$ is orthogonal if any of the equivalent conditions hold

1) The rows of $A$ are an ON basis for $\mathbb{R}^n$

2) The collumns of $A$ are an ON basis for $\mathbb{R}^n$

3) $A$ invertible and $A^{-1} = A^{tr}$

Call $O(n) = \{A \in M \mid AA^{tr} = I\}$ the orthogonal group

**Def:** A $n \times n$ $\mathbb{C}$-matrix is called unitary
if any of the following hold

1) The rows of $A$ are an ON basis for $\mathbb{C}^n$
2) The collumns of $A$ are an ON basis for $\mathbb{C}^n$
3) $A$ invertible and $A^{-1} = \overline{A}^{T}$

Cor: Spectral Thm for orthogonal transformations

$T: V \to V$ orthogonal

Then $\exists$ o.n. basis of eigenvectors with

eigenvalues with $|\lambda| = 1$

**Def**: $U(n) = \{ A \in M_{n \times n}(\mathbb{C}) \mid \underline{A^{-1} = \overline{A}^{tr}} \}$

called ____unitary____ grup

Note if $A \in U(n)$

$$\Rightarrow \det(AA^*) = \det(\mathbb{1}_n) = 1$$

$$\begin{array}{c} \det(AA^*) \\ \shortparallel \\ \det(A)\det(A^*) \\ \shortparallel \end{array}$$

$\det(A)\overline{\det(A)} = 1$

$\rightsquigarrow \boxed{\det(A) = \pm 1}$

<u>Def</u>: $SU(n) := \{A \in U(n) \mid \det A = \underline{\ 1\ }\}$

the "<u>special unitary matrices</u>"

WE DID

IT!

פלסטיק

ג'פיל!